COMPLY/CONSTRAIN GAMES OR GAMES WITH A MULLER TWIST

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Abstract

In this paper we call attention to a category of combinatorial games, which we call *comply/constrain games* or *games with a Muller twist*. We say that we put a *Muller twist* on a familiar game if we modify that game so that each move in the familiar game is followed by a constraint on the next player's move. We solve *Odd-or-Even Nim*, which is a variant of *Nim* with the Muller twist that each player specifies whether their opponent's next removal is to be of an odd or even number of objects, solve a generalization of *Tall-or-Short Wyt Queens*, which puts a Muller twist on *Wyt Queens*, report some results for *Fibonacci-or-Not Nim*, which is a take-away game in which the possible constraints are that the number removed be a Fibonacci number or not, and propose some other games, such as *Nought-or-Cross*.

1. Motivation

The game *Quarto*®, created by Blaise Muller and published by Gigamic, was one of the five Mensa Games of the Year in 1993 and has received other international awards. The sixteen game pieces show all combinations of size (*short* or *tall*), shade (*light* or *dark*), solidity (*shell* or *filled*), and shape (*circle* or *square*). Two players take turns placing pieces on a four by four board and the object is to get four in a line with the same characteristic - all short, for example. Only one piece can go in a cell and, once placed, the pieces stay put. Blaise Muller's brilliant twist is that you choose the piece that your opponent must place and they return the favor after placing it.

We will say that we put a *Muller twist* on a familiar combinatorial game if we modify that game so that each player's move of game pieces in the familiar game is followed by that player's choice of a constraint, from a well defined set of constraints, on the next player's move. We will refer to such a game as a *comply/constrain game*.

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Constraints are common in games. If our opponent in *Tic-Tac-Toe* [3] takes the center square, we cannot play there. However, since that constraint does not involve a choice after the physical move, we do not consider Tic-Tac-Toe in the comply/constrain category of games.

Nor would we call *Fibonacci Nim* (see [1, 5, 13]) a comply/constrain game. (It is traditionally thought of as being played with a single pile of beans, in which the first player can take any number of beans provided they do not take them all. Thereafter the players alternate with the constraint that no player can take more than twice the amount just taken by the previous player.) Although the constraint cannot be deduced by a newcomer observing the pile of beans, the constraint is strictly determined by the previous physical move and thus is not in the comply/constrain category.

A position in a comply/constrain game consists of both a physical configuration of game pieces and a constraint which is not automatic from the current configuration or the move which was made from the previous configuration.

Muller's Quarto® is the first comply/constrain game that we know of and we know of only a few others in the literature.

The second comply and constrain game that we have seen is introduced in the Nov-2001 issue of *The College Mathematics Journal* as Problem 714. The problem posers, A. Holshouser and H. Reiter, called the game *Blocking Nim.* The game is built on *Nim*, which is reviewed in the next section of this paper. Their modification involves a pile-specific restriction on the number that might be removed from each pile. A typical position might be $(7 \textit{ not } 3, 5 \textit{ not } 2, 6 \textit{ not } 6)$ which means that from the pile of 7 we can take any number except 3, and we cannot take exactly 2 from the pile of 5, nor can we take all of the pile of 6. One of the many options for that position is $(7 \textit{ not } 7, 5 \textit{ not } 5, 2 \textit{ not } 1)$

The other comply and constrain games that we know of are due to the same energetic pair, Holshouser and Reiter. One game involves three piles with one configuration blocked. A typical position might be $(5, 4, 3)$ with the restriction that we cannot present the configuration $(5, 4, 1)$. The authors have completely solved the three pile problem. Their paper [10] mentions that the problem with a larger number of piles at the time of writing is still unsolved. Another paper [11] allows blocking several configurations with a single pile. After we submitted the original version of this paper, we found another of their papers [12], in which they consider a composite of subgames with blocking in each subgame.

2. Definitions

If we present our opponent with certain positions and play optimally after that, we are guaranteed to win. Such positions are called $\mathcal{P}$-positions, indicating that the previous player wins if they play optimally for the remainder of the game [1, 9]. If we are ever able to present to our opponent a $\mathcal{P}$-position, our opponent must present to us a non-$\mathcal{P}$-position and, from there, we will be able to present a $\mathcal{P}$-position.

An option is a position that might occur while a move is the position that does occur. We will call an option B of a position A a *child* of A and will refer to A as a *parent* of B.

It is clear that two properties characterize the set of $\mathcal{P}$-positions. A subset C of all possible game positions is the set of $\mathcal{P}$-positions if and only if it satisfies both of the following

$\mathcal{P}$-Properties:

($P1$) *No position in C has a child in C; in other words, no option available to a position in C is an element of C.*

($P2$) *Every position not in C has a child in C; that is, every position not in C has an option in C.*

Obviously, as a special case of ($P2$), the terminal positions are in C, since they are childless positions.

The game of *Nim* involves piles of counters – say piles of beans. Players take turns removing beans. Each player in their turn must take a positive number of beans from one pile of their choosing. The first player that cannot move loses.

It is well known (see [1, 3, 9]) that the $\mathcal{P}$-positions in Nim are easily recognized. Express the pile sizes in binary notation so that each pile size has either a zero or a one in each of the units place, the two's place, the four's place, etc. A position is a $\mathcal{P}$-position in Nim if and only if those binary representations have an even number of ones for each power of two. For example, three piles with 4, 8, and 12 beans have binary representations 100, 1000, and 1100 so there are no one's in the units or twos place and two ones in each of the fours and eights place; therefore three piles with 4, 8, and 12 beans is a $\mathcal{P}$-position in Nim. Succinctly put, the $\mathcal{P}$-positions of Nim are precisely those positions for which the nim-sum (binary sum without carry) of the pile sizes is zero.

3. Odd-or-Even Nim

In this section we put our first Muller twist on classical Nim, introducing *Odd-or-Even Nim*: The first player in Odd-or-Even Nim specifies whether the second player is to take an odd or an even number of beans. Each move after that consists of a comply phase and a constrain phase. In the comply phase, the player gets to choose the pile but must comply with the odd or even restriction on the number of beans removed. In the constrain phase, the player specifies that the opponent must take an odd or even positive number of beans on their next move. A position in this game is of the form $(n_1, \ldots, n_s; even)$ or $(n_1, \ldots, n_s; odd)$, where s is the number of piles, and n_i is the size of the i-th pile.

Theorem 1. *The set of $\mathcal{P}$-positions for Odd-or-Even Nim is the set of positions satisfying one of the following conditions:*

(i) *pile sizes nim-summing to one and constraint of even,*
(ii) *pile sizes nim-summing to zero and constraint of even, or*
(iii) *all pile sizes even and nim-summing to zero and constraint of odd.*

Proof. Denote by C the set satisfying $(i), (ii)$, or (iii) ("C" for "Candidate set"). Let $C.i$ be those elements of C who earn membership by satisfying property (i); define the subsets $C.ii$ and $C.iii$ in a similar manner.

We will show that C satisfies both properties of $\mathcal{P}$-positions. For the first property, $(P1)$, we will consider two cases – first the children of positions in $C.i$ and $C.ii$ and then the children of $C.iii$.

To see that no child of a position in C can be in C, note that the nim-sum of every position in C is zero or one. Any position in C must have pile sizes such that we have an even parity in the twos place and the fours place and the eights place, etc. If the constraint is that an even number is to be removed, the parity must be switched in at least one of those places so no child of a position in $C.i$ or $C.ii$ can be in C.

Since each position in $C.iii$ involves all pile sizes even as well as a constraint of odd and a nim-sum of zero, we have all zeros in the units place and an even number of zeros in the other places. We must "borrow" in order to remove an odd number. This "borrowing" causes at least one parity switch in the other places and thus makes a nim-sum of zero or one impossible. Therefore no child of a position in $C.iii$ can be in C.

To prove $(P2)$, we will consider three cases – even constraint, odd constraint with nonzero nim-sum, odd constraint with zero nim-sum and at least one pile size odd.

Consider being faced with a position with an even constraint that is not in C. Its nim-sum cannot be zero or one. We can make be zero all of the column sums except possibly the last. We can then constrain the next move to be even and present a member of $C.i$ or $C.ii$.

Consider now being faced with a position with an odd constraint that is not in C. Then either the nim-sum is nonzero or not all of the pile sizes are even. In the case of the Nim-sum being non-zero, we can make all places except perhaps the last have even parity and thus present a member of $C.i$ or $C.ii$.

The only type of position in the complement of C which has not been considered has an odd constraint, a zero nim-sum, and at least one pile size odd. If presented with such a position, we can remove a single bean from one of the odd sized piles and thus produce a position with an even parity in every place except possibly the last; thus we can present a position with a nim-sum of zero or one and constrain their move to be even. Therefore, every position not in C has a child in C.

This proves that the elements of C are the $\mathcal{P}$-positions. □

4. Fibonacci-or-Not Nim

In Odd-or-Even Nim we partitioned the natural numbers into the set of odds and its complement. Any set of natural numbers yields a partition which yields such a game. We considered sets determined by the range of functions defined on the set of natural numbers with positive integer values; the constraint set for each game consisted of just two elements – either the player must pick a number in the range of f or the player cannot. We concentrated on functions whose values have a unique representation for the set of natural numbers as sums of values of f, which will always happen if f satisfies, for instance $f(k+1) \leq 2f(k)$.

One example of *f-or-not Nim* that we studied involves $f(n) = F_{n+1}$ where F_n is the n-th Fibonacci number. In particular, *One-Pile Fibonacci-or-Not Nim* is a single pile take-away game in which the constraint is either to remove a Fibonacci number of beans or to remove a non-Fibonacci number of beans.

One approach to finding $\mathcal{P}$-positions is similar to the Sieve of Eratosthenes. Consider the terminal position of having three beans and being constrained to take a non-Fibonacci number of beans. This position, denoted by $(3; not)$, is clearly a $\mathcal{P}$-position and thus any possible parent is a non-$\mathcal{P}$-position. We can immediately mark the following as non-$\mathcal{P}$-positions: $(4; Fib)$, $(5; Fib)$, $(6; Fib)$, $(7; not)$, $(8; Fib)$, $(9; not)$, etc. So we can start with 0 beans and sieve out positions of each constraint (including $(144; Fib)$ and $(145; not)$); then go to 1 bean (since $(1; Fib)$ had not been sieved out) and sieve out some more. We continue with increasing pile size. Those positions not sieved out are $\mathcal{P}$-positions.

A similar approach looks at possible children rather than possible parents. For example, once we have the $\mathcal{P}$-positions for up to eleven beans, we can determine that $(12; Fib)$ is a $\mathcal{P}$-position by verifying that none of its children are $\mathcal{P}$-positions.

We list here the pile sizes up to 500 which, with the appropriate constraints, give a $\mathcal{P}$-position for Fibonacci-or-Not Nim. This solves the game up to 500 beans. It would be interesting if we could give some simple rule for determining whether or not $(n; Fib)$ was a $\mathcal{P}$-position for general n.

0	1	2	3	12	18	27	38	42	49	53	60	64	71
75	86	95	102	106	112	118	122	128	132	148	154	158	165
172	176	190	194	200	212	216	222	226	232	238	242	248	252
258	264	268	274	278	284	288	294	300	306	310	317	324	336
342	346	352	362	368	374	384	388	394	400	410	414	420	424
436	440	446	456	460	466	476	482	488	492	498			

We list here a few winning positions in lexicographical order for *Two-Pile Fibonacci-or-Not Nim* with the understanding that if (n, p) is a winning position then (p, n) is also a winning position with the same constraint. The only $\mathcal{P}$-positions with a constraint of “$not - Fibonacci$” up to 150×150 are:

(0, 0)	(0, 1)	(0, 2)	(0, 3)	(1, 1)	(1, 2)	(1, 3)
(2, 2)	(2, 3)	(3, 3)	(4, 4)	(4, 5)	(4, 6)	(4, 7)
(5, 5)	(5, 6)	(5, 7)	(6, 6)	(6, 7)	(7, 7)	(8, 8)
(8, 9)	(8, 10)	(8, 11)	(9, 9)	(9, 10)	(9, 11)	(10, 10)
(10, 11)	(11, 11)	(12, 13)	(13, 13)	(13, 14)	(13, 15)	(14, 14)
(14, 15)	(15, 15)	(17, 22)	(18, 21)			

and the $\mathcal{P}$-positions with constraint “*Fibonacci*” up to 20×50 are:

(0, 0)	(0, 12)	(0, 18)	(0, 27)	(0, 38)	(0, 42)	(0, 49)	(1, 17)	(1, 26)
(1, 32)	(1, 41)	(1, 48)	(2, 19)	(2, 25)	(2, 29)	(2, 39)	(2, 43)	(3, 20)
(3, 30)	(3, 40)	(3, 44)	(3, 50)	(4, 4)	(4, 16)	(4, 22)	(4, 31)	(4, 42)
(4, 46)	(5, 21)	(5, 32)	(5, 36)	(5, 47)	(6, 23)	(6, 29)	(6, 33)	(6, 43)
(6, 49)	(7, 24)	(7, 30)	(7, 34)	(7, 44)	(7, 48)	(8, 8)	(8, 25)	(8, 35)
(8, 39)	(9, 20)	(9, 27)	(9, 36)	(9, 47)	(10, 26)	(10, 33)	(10, 37)	(10, 49)
(11, 28)	(11, 34)	(11, 38)	(11, 48)	(12, 12)	(12, 19)	(12, 23)	(12, 29)	(12, 35)
(12, 39)	(12, 45)	(13, 24)	(13, 30)	(13, 40)	(13, 44)	(13, 50)	(14, 25)	(14, 31)
(14, 37)	(15, 32)	(15, 38)	(15, 42)	(16, 16)	(16, 22)	(16, 26)	(16, 33)	(16, 45)
(16, 49)	(17, 17)	(17, 21)	(17, 28)	(17, 40)	(17, 44)	(17, 50)	(18, 18)	(18, 25)
(18, 35)	(18, 41)	(19, 19)	(19, 30)	(19, 39)	(20, 20)	(20, 26)	(20, 36)	(20, 46)

It is natural to conjecture that the $\mathcal{P}$-positions with constraint “$not - Fibonacci$” involve relatively few beans while $\mathcal{P}$-positions with constraint “*Fibonacci*” can involve arbitrarily large numbers of beans. There is an intriguing amount of near-regularity in the $\mathcal{P}$-positions but Fibonacci-or-Not Nim has yet to be solved.

5. Tall-or-Short Wyt Queens

The game pieces in *Wyt Queens* all glide in a queenly manner to one corner of the chessboard. If they all glide to the northwest corner of the chess board then each journey must be one of: due north, due west, or due northwest along a diagonal. The queens have the ghost-like qualities that they can pass through each other and several can occupy the same cell at the same time. Just as you would expect, each player in their turn, persuades the queen of their choice to glide a non-zero number of cells and the first player that cannot move loses.

It turns out that anyone who can play Nim well can quickly learn to play Wyt Queens well. A queen on a certain cell is equivalent to a certain size pile in Nim, called the *Sprague-Grundy value.* (Sprague-Grundy theory is fully and beautifully developed in *Winning Ways* [1], *Fair Game* [9], and *On Numbers and Games* [2], as well as some earlier sources.) Once we determine the Sprague-Grundy values of the various cells on the chessboard, we can apply the results of Nim theory – a position is a $\mathcal{P}$-position if and only if the nim-sum of the Sprague-Grundy values is zero.

A queen in the northwest corner is equivalent to a zero sized pile since such a queen is in a terminal position; thus, the Sprague-Grundy value of such a location is zero.

A game position of a queen in one of the two cells sharing a border with that northwest corner cell is equivalent to a nim pile with a single bean since the only option from such a position is to glide to a position which is equivalent to no beans.

Similarly, a game position of a single queen two cells below the northwest corner is equivalent to two beans since it has options equivalent to zero beans or one bean. Another position equivalent to two beans is the location immediately southeast of the northwest corner since it has options of zero (glide like a bishop) and one (glide like a rook).

The cell adjacent to the previous two cells that we have discussed is very interesting. The only options are 1 and 2 so 0 is not an option from this cell. However, from any of its options, we can move to a zero cell. This is called a *non-terminal* zero position.

A non-terminal zero position acts just like an empty pile of beans except that there might be a finite number of moves, each of which can be "reversed", until we reach a terminal zero.

We can continue in this manner and assign an equivalent heap size to each of the cells. It turns out that a single queen in the southeast corner of a standard eight by eight chessboard is equivalent to a nim heap of five beans.

The Sprague-Grundy value of a position in Wyt Queens is the Nim-sum of the Sprague-Grundy values of the locations of the queens. We have discussed six cells so far and the locations are equivalent to pile sizes $0, 1, 2, 2, 0, 5$. The nim-sum expressed in binary is 100 and removing four from the pile of five will produce a position with a nim-sum of zero. We know that we can do this since the Wyt Queen in the southeast corner must be able to glide to some cell with Sprague-Grundy value one because to have a value of five it must have one as an option.

The game of *Tall-or-Short Wyt Queens* involves game pieces of two heights – say three tall and five short Wyt Queens. There are two elements of the constraint set – "She who glides must be tall." and "A short must glide.".

We can think of playing Tall-or-Short Wyt Queens as two subgames and, in our turn, we are constrained to make a single move in two subgame specified by our opponent and we are to constrain our opponent's next move to one of the two subgames. You may prefer a greater variety of queens, say blonds, redheads, etc., so our theorem will concern k subgames.

Definition 2. *Given k ordinary impartial games $G_1, \ldots, G_k$, the forced-subgame sum is the comply/constrain variation where the constraint consists of forcing your opponent to play in a specific game.*

Since it is easily verified that the Sprague-Grundy values for the Wyt Queens have the property that no position with value one has only options with non-terminal zero Sprague-Grundy values, we can use the following theorem to play well the game of Tall-or-Short Queens.

Theorem 3. *If none of the subgames of $G_1, \ldots, G_k$ have a reachable position of Sprague-Grundy value 1 whose only options are non-terminal subgames of value 0, then the $\mathcal{P}$-positions in the forced-subgame sum of $G_1, \ldots, G_k$ all have the constraint in a subgame of value 0 and are of two types:*
(i) Positions in which the constraint is to play in a terminal game.
(ii) Positions in which the total number of non-terminal subgames of value 0 is odd, and there are no terminal subgames.

Proof. Let C be the set of positions that the theorem claims forms the set of $\mathcal{P}$-positions, and let $C.i$ be the set of positions that are in C by virtue of case (i) above and $C.ii$ be the set of positions in C in case (ii) above.

The proof will involve four steps. The first two, in which we will show that $(P1)$ holds, are: the children of positions in $C.i$ cannot have a child in C, nor can the children of $C.ii$.

Positions in $C.i$ are terminal, and thus have no children in C.

If we present a member of $C.ii$, the only children will have a nonzero value for the subgame that we constrained our opponent to. The opponent will thus have to present a position with an even number of subgames with zero value. None of those zero values are terminal. Thus, positions in $C.ii$ have no children in C.

In the remaining two steps, in which we show that $(P2)$ holds, we consider the ways in which we could be presented with a position in the complement of C: Either we have a nonzero value in the game that we are constrained to or we have a non-terminal zero in that subgame and an odd number of other zero values.

Assume that we are constrained to play in a subgame with a value 1. Then we can move to a terminal 0, and constrain our opponent to play in the same game. (This is why we needed the assumption that no subgame has a value of one whose only children have non-terminal zero values.) If we are constrained to play in a subgame with a value $\neq 1$, we can either reply with a zero value in that subgame or to reply with a nonzero value in that subgame, in order to present an odd number of zero values. In either case, we will reply constraining to a subgame with a zero value. If we can reply with an terminal zero in any subgame, we should constrain to that subgame; otherwise, we make that subgame be zero or not in order to present an odd number of zeros.

If we are presented with a position in which we must play in a subgame with a non-terminal zero value and an odd number of other zero values, we should constrain to one of the odd number of zero valued subgames. □

White Knights [1] must always approach the northwest corner of the chessboard. We cannot use the theorem above to play *Tall-or-Short White Knights* since it is easily verified that there are locations with value one whose only options are non-terminal zeros.

In *Non-sovereign Wyt Queens* we point the queen just moved to face north or west or northwest. The next player to move that queen must move in the direction that the queen is pointed. This is not a comply/constrain game since the constraint is not necessarily on the opponent's next move. By the work of Holshouser and Reiter [12], Non-sovereign Wyt Queens can be solved by constructing a dictionary of Sprague-Grundy values associated with each position, where a position is a pair of a cell and a direction.

6. More Unsolved Games with a Muller Twist

Any classical game where a move involves placing a piece of some type and attempting to create some configuration can be given a Muller twist by constraining the piece type to be used. For example, *Nought-or-Cross* is Tic-Tac-Toe played on a three by three board [1] with the twist that we specify whether the opponent is to next place a nought or a cross; the first to get three in a line of the same symbol wins. Similar examples are Tic-Tac-Toe on a 4 by 4 by 4 board [6], on other graphs [4, 7, 8], and Toe-Tac-Tic [3].

If we applied that particular Muller twist to Chess, each player would specify the color of the piece to be moved next. If I tell you to move a white piece and you put either king in checkmate, then you win by constraining me to move that king.

Another type of Muller twist on Chess would keep the concept of ownership of the pieces but would put a constraint on the color of the next cell a move is made to. In Dots and Boxes [1] we might constrain the first line of the next move to be vertical or horizontal. Moves in Checkers could be constrained so that the first piece movement of a turn be slanted left or slanted right.

Recall that Mullers Quarto® pieces have four attributes, such as height, and each attribute has two values, such as short or tall. Each of the sixteen Quarto® pieces could be a Wyt Queen on an eight by eight chessboard, with one of the possible constraints being "She who glides must be square."; seven other constraints would be based on other values of Quarto® attributes (short, tall, round,...). It is easier to name such a game *Quarto® Value Wyt Queens* than to solve it.

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